Ringel-Hall Algebras of Duplicated Tame Hereditary Algebras*

Hongchang Dong, Shunhua Zhang

School of Mathematics, Shandong University, Jinan, 250100, P.R.China

Abstract. Let A be a tame hereditary algebra over a finite field k with q elements, and $\overline{A}$ be the duplicated algebra of A. In this paper, we investigate the structure of Ringel-Hall algebra $\mathscr{H}(\overline{A})$ and of the corresponding composition algebra $\mathscr{C}(\overline{A})$. As an application, we prove the existence of Hall polynomials g_{XY}^{M} for any $\overline{A}$-modules M, X and Y with X and Y indecomposable if A is a tame quiver k-algebra, then we also obtain some Lie subalgebras induced by $\overline{A}$.

Key words: duplicated algebra; Ringel-Hall algebra; Hall polynomial; Lie subalgebra

1 Introduction

The duplicated algebras are interesting algebras that have been introduced recently in the context of cluster categories. In particular, it is the interesting theory of

MSC(2000): 16G10, 17B37

*Supported by the NSF of China (Grant No. 10771112) and NSF of Shandong Province (Grant No. Y2008A05).

Email addresses: hcdong@mail.sdu.edu.cn(H.Dong), shzhang@sdu.edu.cn(S.Zhang)

these algebras which is relevant for the connection to cluster categories, we refer to [ABST1] for details.

Ringel-Hall algebras of finitary rings were introduced by Ringel [R2, R3] in order to deal with possible filtrations of modules with fixed factors. It turns out that Ringel-Hall algebra approach provides a nice framework for the realization of quantized enveloping algebras and Kac-Moody algebras, see [Lu, G, PX, R3-R5]. Later, some fundamental structures for Ringel-Hall algebras of hereditary algebras were proved, see, e.g., [PZ1, PZ2, ZZ, SZ1, SZ2].

In this paper, we investigate the structure of Ringel-Hall algebras of duplicated tame hereditary algebras in section 3, and in section 4 we prove the existence of Hall polynomials g_{XY}^{M} for any $\overline{A}$-modules M, X and Y with X and Y indecomposable (Theorem 4.8) when A is a tame quiver algebra. As an application, in section 5 we also obtain some Lie subalgebras induced by duplicated tame quiver algebras. Section 2 is devoted to some notations and definitions needed for our research.

2 Preliminaries

Let A be a finite dimensional algebra over a field k. We denote by A-mod the category of finitely generated left A-modules, and A-ind a full subcategory of A-mod containing exactly one representative of each isomorphism class of indecomposable A-modules. Given a class $\mathcal{C}$ of A-modules, we denote by add $\mathcal{C}$ the subcategory of A-mod whose objects are the direct summands of finite direct sums of modules in $\mathcal{C}$. We denote by Γ_A the Auslander-Reiten quiver of A and by τ the Auslander-Reiten translation of A. We refer to [ARS, DR, R1] for further notations and definitions in representation theory.

Let M and N be indecomposable A-modules. A path from M to N in A-ind is a sequence of non-zero morphisms

$$M = M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} \cdots \xrightarrow{f_t} M_t = N$$

with all M_i in A-ind. Following [R1], we denote the existence of such a path by

$M \leq N$. We say that M is a predecessor of N (or that N is a successor of M).

More generally, if S_1 and S_2 are two sets of modules, we write $S_1 \leq S_2$ if every module in S_2 has a predecessor in S_1, every module in S_1 has a successor in S_2, no module in S_2 has a successor in S_1 and no module in S_1 has a predecessor in S_2. The notation $S_1 < S_2$ stands for $S_1 \leq S_2$ and $S_1 \cap S_2 = \emptyset$.

Given a finite set M, we denote its cardinality by $|M|$. In the sequel, we always assume that k is a finite field with q elements, that is $|k| = q$, and assume that A is a finite-dimensional tame hereditary algebra over k.

Let M, N_1, N_2 be finite dimensional A-modules. We denote by $G^M_{N_1 N_2}$ the number of submodules L of M with the property that $L \simeq N_2$ and $M/L \simeq N_1$. By [R2] the Ringel-Hall algebra $\mathscr{H}(A)$ is a free abelian group with a basis $\{u_{[M]}\}_{[M]}$ indexed by the isomorphism classes of finite (left) A-modules with the multiplication defined by

$$u_{[N_1]} \cdot u_{[N_2]} = \sum_{[M]} G^M_{N_1 N_2} u_{[M]}$$

Note that we only deal with finite sum since A is a finite ring. We denote by $\mathscr{C}(A)$ the subalgebra of $\mathscr{H}(A)$ generated by simple A-modules which is called composition algebra.

From now on, we always assume that A is a tame hereditary k-algebra and that $\overline{A}$ is the duplicated algebra of A, see [ABST1]. Then $\overline{A} = \begin{pmatrix} A & 0 \\ DA & A \end{pmatrix}$ is the matrix algebra, we see that $\overline{A}$ contains two copies of A given respectively by $e\overline{A}e$ and by $e'\overline{A}e'$, where $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, and $e' = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. We denote the first one by $A = A_0$ and the second one by $A' = A_1$. Accordingly, Q'_A denotes the quiver of A', x' the vertex of Q'_A cerresponding to $x \in (Q_A)_0$, and e'_x the corresponding idempotent. Let S_x, P_x, I_x denote respectively the corresponding simple, indecomposable projective and indecomposable injective module in $\overline{A}$-mod corresponding to $x \in (Q_A \bigcup Q'_A)_0$. For simpleness, we write $Q_A = \{1, \cdots, n\}$ and $Q'_A = \{1', \cdots, n'\}$.

Recall from [ABST2] that the Auslander-Reiten quiver of $\Gamma_{\overline{A}}$ can be described

as follows. It starts with the Auslander-Reiten quiver of $A_0 = A$. Then projective-injective modules start to appear, such projective-injective module has its socle corresponding to a simple A_0-module, and its top corresponding to a simple A_1-module. Next occurs a part denoted by A_{01}-ind where indecomposables contain at same time simple composition factors from simple A_0-modules, and simple composition factors from simple A_1-modules. When all projective-injective modules whose socle corresponding to simple A_0-modules have appeared, we reach the projective A_1-modules and thus the Auslander-Reiten quiver of A_1.

From the description above, we can divide the Auslander-Reiten-quiver $\Gamma_{\overline{A}}$ into 7 parts, denoted by $\mathscr{P}_0$, $\mathscr{R}_0$, $\mathscr{X}_0$, $\mathscr{R}_{01}$, $\mathscr{X}_1$, $\mathscr{R}_1$, $\mathscr{I}_1$ respectively, where $\mathscr{P}_0$ (resp. $\mathscr{I}_1$)is the preprojective (resp. preinjective) component of Γ_{A_0} (resp. Γ_{A_1}); $\mathscr{X}_0$ and $\mathscr{X}_1$ are forms of translation quiver of $\mathbb{Z}Q_A$; $\mathscr{R}_0$, $\mathscr{R}_{01}$ and $\mathscr{R}_1$ are the same types of tubes since A is tame type.

Let M be an $\overline{A}$-module. We denote by $\Omega^{-i}(M)$ the i^{th} cosyzygy of M and by $\Omega^{i}(M)$ the i^{th} syzygy of M respectively. Let $\mathcal{L}_{\overline{A}}$ be the left part of $\overline{A}$-mod. By definitions in [HRS], $\mathcal{L}_{\overline{A}}$ is the full subcategory of $\overline{A}$-mod consisting of all indecomposable $\overline{A}$-modules such that if L is a predecessor of M, then the projective dimension pd L of L is at most one.

It is well known that gl.dim$\overline{A}$, the global dimension of $\overline{A}$, is 3. We denote by Σ_0 the set of all non-isomorphic indecomposable projective A_0-modules, and write $\Sigma_k = \Omega^{-k}\,\Sigma_0 = \{\Omega^{-k}X \mid X \in \Sigma_0\}$ for $1 \leq k \leq 2$.

3 Ringel-Hall algebras of duplicated tame hereditary algebras

In this section, we mainly investigate the structure of Ringel-Hall algebra of duplicated tame hereditary algebras, the decomposition of the composition algebra, indecomposable elements in the composition algebra, and prove that the exceptional elements can be written as skew communicators.

Let A_{01}-ind be the indecomposable $\overline{A}$-modules with the composition factors having both simple A_0-modules and A_1-modules. That is $\overline{A}$-ind $= A_0$-ind $\bigcup A_{01}$-ind $\bigcup A_1$-ind. Note that add(A_{01}-ind) is an exact category which is closed under extensions, we can define the corresponding Ringel-Hall algebra which is denoted by $\mathscr{H}(A_{01})$.

Theorem 3.1. $\mathscr{H}(\overline{A}) = \mathscr{H}(A_0)\mathscr{H}(A_{01})\mathscr{H}(A_1)$.

Proof. Let M be an $\overline{A}$-module. Suppose that $[M] = [M_0] \oplus [M_{01}] \oplus [M_1]$, where $M_{01} \in \text{add } (A_{01} - \text{ind})$, and $M_i \in A_i - \text{mod}$ with $i = 0, 1$.

It is easy to see that $g^{M_{01}\oplus M_1}_{M_{01}\ M_1} = 1$, $g^{M}_{M_0\ M_{01}\oplus M_1} = 1$, and

$$u_{[M_0]}u_{[M_{01}]}u_{[M_1]} = u_{[M_0]}u_{[M_{01}]\oplus[M_1]} = u_{[M_0]\oplus[M_{01}]\oplus[M_1]} = u_{[M]}.$$

□

Let $\mathscr{C}(A_{01}) = \mathscr{C}(\overline{A}) \bigcap \mathscr{H}(A_{01})$ and $\mathscr{C}(A_0)$ (resp. $\mathscr{C}(A_1)$) be the subalgebra of $\mathscr{C}(\overline{A})$ generated by the simple modules $S_1, \cdots, S_n$ (resp. $S_{1'}, \cdots, S_{n'}$). Note that $\mathscr{C}(A_0) = \mathscr{H}(A_0) \bigcap \mathscr{C}(\overline{A})$ and that $\mathscr{C}(A_1) = \mathscr{H}(A_1) \bigcap \mathscr{C}(\overline{A})$, by using Theorem 3.1, we have the following.

Corollary 3.2. $\mathscr{C}(\overline{A}) = \mathscr{C}(A_0)\mathscr{C}(A_{01})\mathscr{C}(A_1)$.

Let M be an indecomposable $\overline{A}$-module. M is said to be exceptional if $\text{Ext}^i_{\overline{A}}(M, M) = 0$ for all $i > 0$.

Theorem 3.3. *Let $\overline{A}$ be the duplicated tame hereditary algebra A and M be an indecomposable $\overline{A}$-module. Then $u_{[M]} \in \mathscr{C}(\overline{A})$ if and only if M is an exceptional $\overline{A}$-module.*

Proof. Assume that M is an exceptional $\overline{A}$-module.

Case I. If $M \in A_0 - \text{ind}$ or $M \in A_1 - \text{ind}$, then $u_{[M]} \in \mathscr{C}(\overline{A})$ by [PZ1, SZ2].

Case II. Assume that $M \in A_{01} - \text{ind}$. First of all, we suppose that M is a projective-injective $\overline{A}$-module.

If $M \in \mathcal{L}_{\overline{A}}$, then top$M = S_{i'}$ and rad$M \in A_0 - \text{mod}$. Note that rad$M$ is a

preinjectve A_0-module and $u_{[\mathrm{rad}M]} \in \mathscr{C}(\overline{A})$, we have the following:

$$u_{[M]} = u_{[S_{i'}]}u_{[\mathrm{rad}M]} - u_{[\mathrm{rad}M]}u_{[S_{i'}]} \in \mathscr{C}(\overline{A}).$$

If $M \notin \mathcal{L}_A$, then $\mathrm{Soc}M = S_i$ and $M/\mathrm{Soc}M \in A_1 - \mathrm{mod}$. In this case, one can easy to see that $M/\mathrm{Soc}M$ is a preprojectve A_1-module and $u_{[M/\mathrm{Soc}M]} \in \mathscr{C}(\overline{A})$, we have the following:

$$u_{[M]} = u_{[M/\mathrm{Soc}M]}u_{[S_i]} - u_{]S_i]}u_{]M/\mathrm{Soc}M]} \in \mathscr{C}(\overline{A}).$$

Finally, we can assume that $M \in A_{01} - \mathrm{ind}$ and M is not a projective-injective $\overline{A}$-module. Read from the Auslander-Reiten quiver of $\overline{A}$ and by using Theorem 9.1 in [PZ1] and Theorem 1 in [SZ2], we know that $u_{[M]} \in \mathscr{C}(\overline{A})$.

Conversely, let M be an indecomposable $\overline{A}$-module and $u_{[M]} \in \mathscr{C}(\overline{A})$. We want to prove that M is an exceptional $\overline{A}$-module. If $M \in A_0 - \mathrm{ind}$ or $M \in A_1 - \mathrm{ind}$, then M is an exceptional $\overline{A}$-module follows from [ZZ] if A is a tame quiver algebra and follows from [SZ2] when A is a non-simply-laced tame hereditary algebra.

Now assume that $M \in A_{01} - \mathrm{ind}$ and we may assume that M is not a projective-injective $\overline{A}$-module. It is easy to read from the Auslander-Reiten quiver of $\overline{A}$ that M is in a full subquiver of $\Gamma_{\overline{A}}$ which is isomorphic to $\Gamma_{D(A_0)}$. Then M is an exceptional $\overline{A}$-module follows from [ZZ, SZ2] again. This completes the proof. $\square$

Example 3.4. Let $\overline{A}$ be the duplicated tame quiver algebra of type $\widetilde{D}_4$. That is, $\overline{A} = k\overline{\widetilde{D}_4}/I$, and $\overline{\widetilde{D}_4}$ is the following quiver,

$$\overline{\widetilde{D}_4} : \begin{array}{ccccccc} & & 2 & & & & 2' \\ & \swarrow & & \nwarrow & & \swarrow & \\ 1 & \leftleftarrows & \begin{array}{c}3\\4\end{array} & \leftleftarrows & 1' & \leftleftarrows & \begin{array}{c}3'\\4'\end{array} \\ & \nwarrow & & \swarrow & & \nwarrow & \\ & & 5 & & & & 5' \end{array}.$$

Then the indecomposable projective-injective $\overline{A}$-modules are represented by their Loewy series as the following,

$$P'_1 = \begin{array}{c}1'\\2345\\1\end{array},\ P'_2 = \begin{array}{c}2'\\1'\\2\end{array},\ P'_3 = \begin{array}{c}3'\\1'\\3\end{array},\ P'_4 = \begin{array}{c}4'\\1'\\4\end{array},\ P'_5 = \begin{array}{c}5'\\1'\\5\end{array}.$$

We should mention that the minimal positive imaginary root of $k\widetilde{D}_4$ is $\delta = (2,1,1,1,1)$ and every indecomposable $\overline{A}$-module M which belongs to $\mathscr{R}_0$, $\mathscr{R}_{01}$ or $\mathscr{R}_1$ with $l(M) \geq 6$ is not exceptional, where $l(M)$ is the length of M.

According to Theorem 3.3, for any indecomposable $\overline{A}$-module M, $u_{[M]}$ belongs to $\mathscr{C}(\overline{A})$ if and only if M belongs to $\mathscr{P}_0$, $\mathscr{X}_0$, $\mathscr{X}_1$, $\mathscr{I}_1$ or to $\mathscr{R}_0$, $\mathscr{R}_{01}$, $\mathscr{R}_1$ with $l(M) < 6$.

The following concept is defined in [PZ2]. Let B be a k-algebra, and $x, y \in B$, and $c, d \in k^* = k \backslash \{0\}$. The element $cxy - dyx$ is called a *skew commutator* of x and y. Let $X = \{x_1, \cdots, x_n\}$ be a set of B. Define the sets X_i inductively: Let $X_0 = X$. Let X_i be the set of all skew commutators of arbitrary two different elements in $\bigcup\limits_{j<i} X_j$. An element $x \in B$ is called an iterated skew commutator of $x_1, \cdots, x_n$, provided that there exists a positive integer m such that $x \in X_m$.

The following theorem indicates that an indecomposable non-simple $\overline{A}$-module which belongs to $\mathscr{C}(\overline{A})$ can be written as an iterated skew commutator of simple $\overline{A}$-modules.

Theorem 3.5. *Let A be a tame hereditary algebra over k and $\overline{A}$ be the duplicated algebra of A. Let M be a non-simple indecomposable $\overline{A}$-module. Then the element $u_{[M]} \in \mathscr{C}(\overline{A})$ can be written as an iterated skew commutator of the isoclasses of simple $\overline{A}$-modules.*

Proof. According to Theorem 3.3, we know that M is an exceptional $\overline{A}$-module.

If $M \in A_0 - \mathrm{ind}$ or $M \in A_1 - \mathrm{ind}$, then M is an iterated skew commutator of the isoclasses of simple $\overline{A}$-modules by Theorem 2.1 in [PZ2].

Now, Let $M \in A_{01} - \mathrm{ind}$ and M be a projective-injective $\overline{A}$-module.

If $M \in \mathcal{L}_{\overline{A}}$, according to the proof of Theorem 3.3, we can write $u_{[M]}$ as following:

$$u_{[M]} = u_{[S_{i'}]} u_{[\mathrm{rad} M]} - u_{[\mathrm{rad} M]} u_{[S_{i'}]} \in \mathscr{C}(\overline{A}),$$

$\mathrm{rad} M \in A_0 - \mathrm{mod}$ is a preinjectve A_0-module and $\mathrm{top} M = S_{i'}$. By using the Theorem 2.1 in [PZ2], $u_{[\mathrm{rad} M]}$ is an iterated skew commutator of the isoclasses of simple A_0-modules, hence $u_{[M]}$ can be written as an iterated skew commutator of the isoclasses of simple $\overline{A}$-modules.

If $M \notin \mathcal{L}_{\overline{A}}$, then $\mathrm{Soc}M = S_i$ is a simple A_0-module and $M/\mathrm{Soc}M$ is a preprojectve A_1-module. By using Theorem 2.1 in [PZ2] again, $u_{[M/\mathrm{Soc}M]}$ can be written as an iterated skew commutator of the $S_{1'}, \cdots, S_{n'}$.

Note that $u_{[M]} = u_{[M/\mathrm{Soc}M]}u_{[S_i]} - u_{[S_i]}u_{[M/\mathrm{Soc}M]} \in \mathscr{C}(\overline{A})$, thus $u_{[M]}$ is an iterated skew commutator of the isoclasses of simple $\overline{A}$-modules.

Finally, we can assume that $M \in A_{01} - \mathrm{ind}$ and M is not a projective-injective $\overline{A}$-module. It follows from the Auslander-Reiten quiver of $\overline{A}$ that M is in a full subquiver of $\Gamma_{\overline{A}}$ which is isomorphic to $\Gamma_{D(A_0)}$. Since M is an exceptional $\overline{A}$-module, we know that $u_{[M]}$ is an iterated skew commutator of the isoclasses of simple $\overline{A}$-modules by using Theorem 2.1 in [PZ2]. The proof is completed. $\square$

4 Some Hall polynomials for duplicated tame hereditary algebras

In this section, we always assume that A is a tame quiver algebra over k and $\overline{A}$ be the duplicated algebra of A, and we will prove that some Hall polynomials for duplicated tame hereditary algebras exist. Note that we can, in this case, divide the Auslander-Reiten quiver $\Gamma_{\overline{A}}$ into 7 parts, denoted by $\mathscr{P}_0$, $\mathscr{R}_0$, $\mathscr{X}_0$, $\mathscr{R}_{01}$, $\mathscr{X}_1$, $\mathscr{R}_1$, $\mathscr{I}_1$ respectively.

Let E be a field extension of k. For any k-space V, we denote by V^E the E-space $V \otimes_k E$; then, of course, A^E naturally becomes an E-algebra. If S is a simple A-module, according to Theorem 7.5 in [La], we know that S^E is the simple A^E-module. For any $M \in A$-mod, E is called M-conservative for A if for any indecomposable summand N of M, $(\mathrm{End}N/\mathrm{rad}\ \mathrm{End}N)^E$ is a field. Under field isomorphism, we put

$$\Omega_M = \{E | E \text{ is a finite field extension of } k \text{ and } E \text{ is } M-\text{conservative for } A\}.$$

Note that Ω_M is an infinite set, since M has only finitely indecomposable summands. By [SZ1], we say that Hall polynomials exist for A, if for any M, N_1, $N_2 \in A$-mod, there exists a polynomial $g^M_{N_1N_2} \in \mathbf{Z}[x]$ and an infinite subset $\Omega^M_{N_1N_2}$

of $\Omega_{M\oplus N_1\oplus N_2}$, such that for any $E \in \Omega^M_{N_1N_2}$,

$$g^M_{N_1N_2}(|E|) = G^{M^E}_{N_1^E N_2^E}.$$

Such a polynomial $g^M_{N_1N_2}$ is called a Hall polynomial of A.

Remark. When A is a representation-finite algebra, the above definition is the same as in [R5].

The following results were proved in [SZ1] for tame quiver algebras, and we observe that they are also true for duplicated tame hereditary algebras, we refer to [SZ1] for details.

Lemma 4.1. *Assume that $u_{[M]} = u_{[M_1]} + u_{[M_2]}$ in $\mathscr{H}(\overline{A})$. Then $G^L_{MN} = G^L_{M_1N} + G^L_{M_2N}$ for any $\overline{A}$-modules L and N.*

Lemma 4.2. *Given M, $N \in \overline{A}$-*mod*, then there exists a nonnegative integer $h(M,N)$ such that $|\mathrm{Hom}_{\overline{A}^E}(M^E, N^E)| = |E|^{h(M,N)}$ for any $E \in \Omega_{M\oplus N}$.*

Lemma 4.3. *Let $N_1, N_2, \cdots, N_t$ be simple $\overline{A}$-modules except at most only one. Then there exists the Hall polynomial $g^M_{N_1N_2\cdots N_t}$ for all $M \in \overline{A}$-*mod*.*

Lemma 4.4. *Let M, N, L be $\overline{A}$-modules with $N \in \mathscr{P}_0$, $\mathscr{X}_0$, $\mathscr{X}_1$ or $\mathscr{I}_1$. Then the Hall polynomials g^M_{NL} and g^M_{LN} exist.*

Proof. By duality, we only need to prove the existence of the Hall polynomial g^M_{NL}. Note that for any indecomposable $\overline{A}$-module $X \in \mathscr{P}_0$, $\mathscr{X}_0$, $\mathscr{X}_1$ or $\mathscr{I}_1$ which is exceptional, according to Theorem 3.3, we know that $u_{[X]} \in \mathscr{C}(\overline{A})$. Therefore we can assume that

$$u_{[N]} = \sum_{i_1,\cdots,i_t \in \{1,\cdots,n,1',\cdots,n'\}} a_{i_1\cdots i_t} u_{[s_{i_1}]} \cdots u_{[s_{i_t}]},$$

where $a_{i_1\cdots i_t} \in \mathbf{Z}$. By using Lemma 4.1, we have

$$G^M_{NL} = \sum_{i_1,\cdots,i_t \in \{1,\cdots,n,1',\cdots,n'\}} a_{i_1\cdots i_t} G^M_{S_{i_1}\cdots S_{i_t}L}.$$

According to Lemma 4.3, we have Hall polynomials $g^M_{S_{i_1}\cdots S_{i_t}L} \in \mathbf{Z}[x]$ such that there exists an infinite subset Ω^M_{NL} of $\Omega_{M\oplus N\oplus L}$ and for any $E \in \Omega^M_{NL}$

$$G^{M^E}_{S^E_{i_1}\cdots S^E_{i_t}L^E} = g^M_{S_{i_1}\cdots S_{i_t}L}(|E|).$$

Let $g^M_{NL} = \sum_{i_1,\cdots,i_t\in\{1,\cdots,n,1',\cdots,n'\}} a_{i_1\cdots i_t} g^M_{S_{i_1}\cdots S_{i_t}L} \in \mathbf{Z}[x]$. For any $E \in \Omega^M_{NL}$, we have that $G^{M^E}_{N^E L^E} = g^M_{NL}(|E|)$, that is, g^M_{NL} is the Hall polynomial of $\overline{A}$. This completes the proof. $\square$

Lemma 4.5. *Let M, N and L be $\overline{A}$-modules with N and L indecomposable. If $N, L \in \mathscr{R}_0$, $N, L \in \mathscr{R}_{01}$ or $N, L \in \mathscr{R}_1$, then the Hall polynomial g^M_{NL} exists.*

Proof. We may assume that M is an extension of L by N, since otherwise we may take $g^M_{NL} = 0$. Therefore we have a short exact sequence $0 \to L \to M \to N \to 0$, it follows that M, N, L belong to the same part of $\Gamma_{\overline{A}}$, that is, $M, N, L \in \mathscr{R}_0$, $M, N, L \in \mathscr{R}_{01}$ or $M, N, L \in \mathscr{R}_1$. By using the same method as Lemma 2.9 in [SZ1], we know that g^M_{NL} exists. This completes the proof. $\square$

For any $M, N, L \in \overline{A} - \mathrm{mod}$, we denote by $\mathrm{Ext}^1_{\overline{A}}(N, L)_M$ the set of all exact sequences in $\mathrm{Ext}^1_{\overline{A}}(N, L)$ with middle term M. The following lemma was proved in [P, Rie].

Lemma 4.6. *For any $\overline{A}$-modules M, N, L,*

$$G^M_{NL} = \frac{|\ \mathrm{Ext}^1_{\overline{A}}(N, L)_M\ | \cdot |\ \mathrm{Aut}_{\overline{A}}M\ |}{|\ \mathrm{Aut}_{\overline{A}}N\ | \cdot |\ \mathrm{Aut}_{\overline{A}}L\ | \cdot |\ \mathrm{Hom}_{\overline{A}}(N, L)\ |}.$$

Lemma 4.7. *Let M, N and L be $\overline{A}$-modules with N and L indecomposable. Assume that $N \in \mathscr{R}_i$, $L \in \mathscr{R}_j$ with $i \neq j \in \{0, 01, 1\}$. Then the Hall polynomial g^M_{NL} exists.*

Proof. We only need to consider the cases $L \in \mathscr{R}_0$ with $N \in \mathscr{R}_{01}$, and $L \in \mathscr{R}_{01}$ with $N \in \mathscr{R}_1$, since in other cases we have that $\mathrm{Ext}^1_{\overline{A}}(N, L) = 0$, and by using Lemma 4.6, the existence of the Hall polynomial g^M_{NL} follows.

Case I. Let $L \in \mathscr{R}_0$ with $N \in \mathscr{R}_{01}$. Assume that $E(L)$ is the injective envelope

of L, then we have a short exact sequence

$$(*) \qquad 0 \to L \to E(L) \to \Omega^{-1}L \to 0,$$

where $E(L)$ is projective-injective $\overline{A}$-module since $L \in \mathscr{R}_0$, and $\Omega^{-1}L$ is an indecomposable $\overline{A}$-module which belongs to $\mathscr{R}_{01}$. Note that $E(L)$ is a predecessor of N, by applying $\mathrm{Hom}_{\overline{A}}(N,-)$ to $(*)$, we obtain that $\mathrm{Hom}_{\overline{A}}(N,\Omega^{-1}L) \simeq \mathrm{Ext}^1_{\overline{A}}(N,L)$. Hence $\dim_k \mathrm{Ext}^1_{\overline{A}}(N,L) = \dim_k \mathrm{Hom}_{\overline{A}}(N,\Omega^{-1}L) \leq 1$ since N and $\Omega^{-1}L$ are indecomposable $\overline{A}$-modules belonging to $\mathscr{R}_{01}$. For any $\overline{A}$-module M, according to Lemma 4.2 and Lemma 4.6 we know that the Hall polynomial g^M_{NL} exists.

Case II. Let $L \in \mathscr{R}_{01}$ with $N \in \mathscr{R}_1$. By using the same method as in Case I, we can prove that the Hall polynomial g^M_{NL} exists. This completes the proof. □

Theorem 4.8. *Let X and Y be indecomposable $\overline{A}$-modules. Then for any $\overline{A}$-module M, there exists the Hall polynomial g^M_{XY}.*

Proof. If one of the indecomposable $\overline{A}$-modules X and Y belongs to $\mathscr{P}_0$, $\mathscr{X}_0$, $\mathscr{X}_1$ or $\mathscr{I}_1$, by Lemma 4.4, we know that the Hall polynomial g^M_{NL} exists.

If none of X and Y belongs to $\mathscr{P}_0$, $\mathscr{X}_0$, $\mathscr{X}_1$ or $\mathscr{I}_1$, then X and Y must belong to $\mathscr{R}_i$ or $\mathscr{R}_j$, where $(i,j \in \{0,01,1\})$. In case $i=j$, then the existence of the Hall polynomial g^M_{XY} follows from Lemma 4.5. If $i \neq j$, according to Lemma 4.7, we have the Hall polynomial g^M_{XY} exists. The proof is completed. □

Remark. If A is a representation-finite hereditary k-algebra, then the duplicated algebra $\overline{A}$ is represented-direct, thus according to [R5], we know that the Hall polynomial g^M_{NL} exists for any $\overline{A}$-modules M, N, L.

5 The Lie subalgebras induced by duplicated tame hereditary algebras

In this section, we also assume that A is a tame quiver algebra over k and $\overline{A}$ is the duplicated algebra of A, and we will investigate some Lie subalgebras induced by $\overline{A}$ which seem to have an independent interest.

Let Ω be an infinite set of finite field extension of k up to isomorphism. Since

A is a tame quiver algebra, according to [CD] and Theorem 7.5 in [La], we know that E is S-conservative for any simple $\overline{A}$-module S.

Denote by $\mathscr{H}(\overline{A},\Omega)$ the subring of $\prod_{E\in\Omega}\mathscr{H}(\overline{A^E})$ generated by $\{([M^E])_{E\in\Omega}|M\in \overline{A}-\text{mod}\}$ and $q_\Omega=(|E|u_{[0]})_{E\in\Omega}$. Denote by $\mathscr{H}(\overline{A})_1$ the quotient ring $\mathscr{H}(\overline{A},\Omega)/(q_\Omega-1)\mathscr{H}(\overline{A},\Omega)$, called the degenerate Ringel-Hall algebra of $\overline{A}$. The subalgebra of $\mathscr{H}(\overline{A})_1$ generated by the simple $\overline{A}$-modules, denoted by $\mathscr{C}(\overline{A})_1$, is called the degenerate composition algebra of $\overline{A}$.

The following Lemma was proved in [R5].

Lemma 5.1. *Let $M,X,Y\in\overline{A}$-mod with X and Y indecomposable. For any $E\in\Omega_{M\oplus X\oplus Y}$, then*

(1) *If $M\not\simeq X\oplus Y$, then $|E|-1$ divides $G^{M^E}_{X^EY^E}$;*

(2) *If $M\simeq X\oplus Y$. If $X\simeq Y$, then $|E|-1$ divides $G^{M^E}_{X^EY^E}-2$;*
If $X\not\simeq Y$, then $|E|-1$ divides $G^{M^E}_{X^EY^E}-1$.

Let $\mathscr{L}(\overline{A})=\bigoplus_{N\in\overline{A}-\text{ind}}\mathbf{Z}u_{[N]}$ be the free Abel group with basis the set of isomorphism classes determined by indecomposable $\overline{A}$-modules.

Theorem 5.2. *$\mathscr{L}(\overline{A})$ is the Lie subalgebra of $\mathscr{H}(\overline{A})_1$.*

Proof. Assume that$X$ and Y are indecomposable $\overline{A}$-modules such that $X\not\simeq Y$. For any $\overline{A}$-module M, according to Theorem 4.8, we know that the Hall polynomials g^M_{XY} and $g^M_{YX}\in\mathbf{Z}[x]$ exist, satisfying for any $E\in\Omega_{M\oplus X\oplus Y}$, $g^M_{XY}(|E|)=G^{M^E}_{X^EY^E}$ and $g^M_{YX}(|E|)=G^{M^E}_{Y^EX^E}$. By Lemma 5.1, in $\mathscr{H}(\overline{A})_1$,

$$u_{[X]}\cdot u_{[Y]}=\sum_{Z\in\overline{A}-\text{ind}}g^Z_{XY}(1)u_{[Z]}+u_{[X\oplus Y]},$$

$$u_{[Y]}\cdot u_{[X]}=\sum_{Z\in\overline{A}-\text{ind}}g^Z_{YX}(1)u_{[Z]}+u_{[X\oplus Y]},$$

therefore

$$[u_{[X]},u_{[Y]}]=\sum_{Z\in\overline{A}-\text{ind}}(g^Z_{XY}(1)-g^Z_{YX}(1))u_{[Z]}\in\mathscr{L}(\overline{A}),$$

so we have that $\mathscr{L}(\overline{A})$ is the Lie subalgebra of $\mathscr{H}(\overline{A})_1$. $\square$

Let $\mathscr{L}'(\overline{A})$ be the Lie subalgebra of $\mathscr{L}(\overline{A})$ generated by the simple $\overline{A}$-modules. According to [R5] and by using PBW-basis Theorem, we have the following.

Proposition 5.3. *$\mathscr{C}(\overline{A})_1 \otimes_{\mathbf{Z}} \mathbf{Q}$ is the universal enveloping algebra of $\mathscr{L}'(\overline{A}) \otimes_{\mathbf{Z}} \mathbf{Q}$.*

Let $\mathscr{L}_0(\overline{A})$ be the Lie subalgebra of $\mathscr{L}'(\overline{A})$ generated by $S_1, \cdots, S_n$ and $\mathscr{L}_1(\overline{A})$ the Lie subalgebra of $\mathscr{L}'(\overline{A})$ generated by $S_{1'}, \cdots, S_{n'}$ respectively. Then by [Rie] we have $\mathscr{L}_0(\overline{A}) \cong \mathscr{L}_1(\overline{A})$ as Lie subalgebras, which is also isomorphic to the positive part of the corresponding affine Kac-Moody algebra of type A.

We denote by Σ_1^{PI} the set of indecomposable projective-injective $\overline{A}$-modules which are predecessors of Σ_1, and by Σ_2^{PI} the set of indecomposable projective-injective $\overline{A}$-modules which are successors of Σ_1. Note that $\Sigma_1 < \Sigma_2^{\mathrm{PI}} < \Sigma_2$.

Let $\Xi_{01} = \{\ X \in \overline{A} - \mathrm{ind} \mid \Sigma_1^{\mathrm{PI}} \leq X \leq \Sigma_2^{\mathrm{PI}}\}$. For any $M \in \Xi_{01}$ which is not projective-injective, reading from the Auslander-Reiten quiver of $\Gamma_{\overline{A}}$, we know that $\Sigma_1 \leq X \leq \tau^{-2}\Sigma_2$. We denote by $\mathscr{L}(\Xi_{01})$ the free subgroup $\bigoplus\limits_{N \in \Xi_{01}} \mathbf{Z}u_{[N]}$ of $\mathscr{L}(\overline{A})$ and by $\mathscr{H}(\Xi_{01})_1$ the subalgebra of $\mathscr{H}(\overline{A})_1$ generated by indecomposable $\overline{A}$-modules in Ξ_{01}. Let $\mathscr{L}_{01}(\overline{A}) = \mathscr{L}(\Xi_{01}) \bigcap \mathscr{C}(\overline{A})_1$ and $\mathscr{C}(\overline{A})_{01} = \mathscr{H}(\Xi_{01})_1 \bigcap \mathscr{C}(\overline{A})_1$.

Theorem 5.4. (1) *$\mathscr{L}_{01}(\overline{A})$ is a Lie subalgebra of $\mathscr{L}'(\overline{A})$. In particular, $\mathscr{L}'(\overline{A}) = \mathscr{L}_0(\overline{A}) \oplus \mathscr{L}_{01}(\overline{A}) \oplus \mathscr{L}_1(\overline{A})$.*

(2) *$\mathscr{C}(\overline{A})_{01} \otimes_{\mathbf{Z}} \mathbf{Q}$ is the universal enveloping algebra of $\mathscr{L}_{01}(\overline{A}) \otimes_{\mathbf{Z}} \mathbf{Q}$.*

Proof: (2) is trivial, so we only need to prove (1). According to the Auslander-Reiten quiver $\Gamma_{\overline{A}}$, we know that add Ξ_{01} is closed under extensions, thus $\mathscr{L}(\Xi_{01})$ is a Lie subalgebra of $\mathscr{L}(\overline{A})$, hence $\mathscr{L}_{01}(\overline{A})$ is a Lie subalgebra of $\mathscr{L}'(\overline{A})$. The proof is completed. $\square$

Let $\delta = (a_1, \cdots, a_n)$ be the minimal positive imaginary root of A and $m = \sum\limits_{i=1}^{n} a_i$.

Theorem 5.5. *Let M be an indecomposable $\overline{A}$-module and $l(M)$ be the length*

of M. Assume that m cannot divide $l(M)$, i.e., $m \not| l(M)$, then $u_{[M]}$ belongs to $\mathscr{C}(\overline{A})_1 \otimes_{\mathbf{Z}} \mathbf{Q}$.

Proof: First we assume that M belongs to one of $\mathscr{P}_0$, $\mathscr{X}_0$, $\mathscr{X}_1$, $\mathscr{I}_1$. If M is not a projective-injective $\overline{A}$-module, then $u_{[M]} \in \mathscr{C}(\overline{A})_1 \otimes_{\mathbf{Z}} \mathbf{Q}$ follows by Theorem 3.2 in [SZ3].

If M is projective-injective, then according to the proof of Theorem 3.3, we know that $u_{[M]} \in \mathscr{C}(\overline{A})_1 \otimes_{\mathbf{Z}} \mathbf{Q}$.

Finally, we assume that M belongs to one of $\mathscr{R}_0$, $\mathscr{R}_{01}$, $\mathscr{R}_1$, that is M belongs to one tube. Note that m cannot divide $l(M)$, it follows that M must belong to non-homogenous tube, then by Corollary 3.1 in [SZ3], we know that $u_{[M]}$ belongs to $\mathscr{C}(\overline{A})_1 \otimes_{\mathbf{Z}} \mathbf{Q}$. The proof is finished. $\square$

Remark. According to Theorem 3.3 and Theorem 5.5, there is a big difference between the indecomposable $\overline{A}$-modules which belong to $\mathscr{C}(\overline{A})$ and those which belong to $\mathscr{C}(\overline{A})_1$, see the following example.

Example 5.6. Let $\overline{A}$ be the duplicated tame quiver algebra of type $\widetilde{D}_4$. That is, $\overline{A} = \overline{k\widetilde{D_4}}/I$ as in Example 3.4. According to Theorem 5.5, $u_{[M]}$ belongs to $\mathscr{C}(\overline{A})_1$ if and only if M belongs to $\mathscr{P}_0$, $\mathscr{X}_0$, $\mathscr{X}_1$, $\mathscr{I}_1$ or to $\mathscr{R}_0$, $\mathscr{R}_{01}$, $\mathscr{R}_1$ with $6 \not| l(M)$.

The following example indicates that the converse of Theorem 5.5 does not hold.

Example 5.7. Let K be the Kronecker algebra and $\overline{K}$ be the duplicated algebra of K. We may assume that $\overline{K} = kQ_{\overline{K}}/I$ with $Q_{\overline{K}} : 1 \leftleftarrows 2 \leftleftarrows 1' \leftleftarrows 2'$. The indecomposable projective-injective $\overline{A}$-modules are $P_{1'} = \begin{matrix} 1' \\ 22 \\ 1 \end{matrix}$ and $P_{2'} = \begin{matrix} 2' \\ 1'1' \\ 2 \end{matrix}$ which are represented by their Loewy series.

Let $\mathscr{C}(\overline{K})_1$ be the degenerated composition algebra generated by simple $\overline{K}$-modules $S_1, S_2, S_{1'}, S_{2'}$. Note that $\delta = (1,1)$ is the minimal positive imaginary root of K and $m = 2$ in this case. $l(P_{1'}) = 4$ and $u_{[P_{1'}]} \in \mathscr{C}(\overline{A})_1 \otimes_{\mathbf{Z}} \mathbf{Q}$ since $u_{[P_{1'}]} = [u_{[S_{1'}]}, [u_{[S_1]}, u_{[S_2]}u_{[S_2]}]]$.

Let M be an indecomposable $\overline{K}$-module. If $l(M)$, the length of M, is a positive

even number, then $u_{[M]} \in \mathscr{C}(\overline{A})_1 \otimes_{\mathbf{Z}} \mathbf{Q}$ if and only if M is $P_{1'}$ or $P_{2'}$ since otherwise M belongs to one of homogeneous tubes and in this case $u_{[M]}$ dose not belong to $\mathscr{C}(\overline{A})_1 \otimes_{\mathbf{Z}} \mathbf{Q}$.

If $l(M)$ is a positive odd number, then from the Auslander-Reiten quiver of $\Gamma_{\overline{K}}$, we know that M belongs to one of components $\mathscr{P}_0$, $\mathscr{X}_0$, $\mathscr{X}_1$, $\mathscr{I}_1$. By using Theorem 3.3, we have that $u_{[M]} \in \mathscr{C}(\overline{A})_1 \otimes_{\mathbf{Z}} \mathbf{Q}$.

Acknowledgments. The authors would like to thank Wenxu Ge and Hongbo Lv for many useful discussions.

References

[ABST1] I.Assem, T.Brüstle, R.Schiffer, G.Todorov, Cluster categories and duplicated algebras. *J. Algebra*, 305(2006), 548-561.

[ABST2] I.Assem, T.Brüstle, R.Schiffer, G.Todorov, m-cluster categories and m-replicated algebras. *Journal of pure and applied Algebra* 212(2008), 884-901.

[ARS] M.Auslander, I.Reiten, S.O.Smalϕ, Representation Theory of Artin Algebras. Cambridge Univ. Press, 1995.

[CD] J.Chen, B.Deng, Fundamental relations in Ringel-Hall algebras. J.Algebra, 320(2008), 1133 -1149.

[DR] V.Dlab, C.M.Ringel, Indecomposable representations of graphs and algebras. Mem.Amer.Math.Soc., 173 (1976)

[G] J.A.Green, Hall algebras,hereditary algebras and quantum groups. Inventiones Math., 120(1995). 361-377.

[HRS] D.Happel, I.Reiten, S.O.Smalϕ, Tilting in abelian categories and quasitilted algebras, Mem.Amer.Math.Soc., 575, 1996.

[La] T.Y.Lam, First Course in Noncommutative Rings. Graduate Texts in Mathematics, Vol. 131, Springer-Verlag, New York, 1991.

[Lu] G.Lusztig, Intoduction to quantum groups. Birkhǎuser. Boston.1993.

[P] L.Peng, Some Hall polynomials for representation-finite trivial extension algebras. J.Algebra, 197(1997),1-13.

[PX] L.Peng, J.Xiao, Triangulated categories and Kac-Moody algebras. Invent. math., 140(2000), 563-603.

[R1] C.M.Ringel, Tame algebras and integral quadratic forms. Springer Lecture Notes in Math., (1099), 1984

[R2] C.M.Ringel, Hall algebra. In: Topics in Algebras. Banach Centre Publ. Warszawa. 1990, 26, 433-447.

[R3] C.M.Ringel, Hall algebras and quantum groups. Inventiones Math., 101 (1990), 583-592

[R4] C.M.Ringel, Green's theorem on Hall algebras. Canad.Math.Soc. Conf.Proc. Vol 19, pp:185-245.

[R5] C.M.Ringel, Lie algebra arising in representation theory, London Math Soc., 168, Cambridge University Press, 1992, 284-291.

[Rie] C.H.Riedtmann, Lie algebras generated by indecomposables. J.Algebra, 170(1994), 526-546.

[PZ1] P.Zhang, Composition algebras of affine type. J.Algebra.206(1998),505-540.

[PZ2] P.Zhang, Rigids as iterated skew commutators of simples. Algebr Represent Theory, 9(2006), 539-555.

[ZZ] P.Zhang, S.Zhang, Indecomposables as elements in affine composition algebras. J.Algebra, 210(1998), 614-629.

[SZ1] S.Zhang, The Hall polynomials for tame quiver algebras. J.Algebra, 239(2001), 606-614.

[SZ2] S.Zhang, Triangular decomposition of tame non-simple-laced composition algebras. J.Algebra, 241(2001),548-577.

[SZ3] S.Zhang, Lie algebra determined by tame hereditary algebras (In chinese). Chinese Ann. Math. 21(5)(A)(2000), 619-621.